

# Bernoulli type polynomials on Umbral Algebra

Rahime Dere and Yilmaz Simsek

Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya,  
Turkey

E-mail: rahimedere@gmail.com and ysimsek@akdeniz.edu.tr

## Abstract

The aim of this paper is to investigate generating functions for modification of the Milne-Thomson's polynomials, which are related to the Bernoulli polynomials and the Hermite polynomials. By applying the Umbral algebra to these generating functions, we provide to deriving identities for these polynomials.

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## 1. INTRODUCTION

Throughout of this paper, we use the following notations:

$$\mathbb{N} := \{1, 2, 3, \dots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

$$\delta_{n,k} = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k, \end{cases}$$

and

$$(x)_b = x(x-1) \dots (x-b+1),$$

where  $b \in \mathbb{N}$ .

Here, we use the notations and definitions which are related to the umbral algebra and calculus cf. [6].

Let  $P$  be the algebra of polynomials in the single variable  $x$  over the field complex numbers. Let  $P^*$  be the vector space of all linear functionals on  $P$ . Let  $\langle L | p(x) \rangle$  be the action of a linear functional  $L$  on a polynomial  $p(x)$ . Let  $F$  denotes the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,$$

cf. [6].

This kind of algebra is called an umbral algebra. Each  $f \in F$  defines a linear functional on  $P$  and for all  $k \geq 0$ ,  $a_k = \langle f(t) | x^k \rangle$ . The order  $o(f(t))$  of a power series  $f(t)$  is the smallest integer  $k$  for which the coefficient of  $t^k$  does not vanish. A series  $f(t)$  for which

$o(f(t)) = 1$  is called a delta series. And a series  $f(t)$  for which  $o(f(t)) = 0$  is called a invertible series cf. [6].

Let  $f(t), g(t)$  be in  $F$ , we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle, \quad (1.1)$$

cf. [6]. For all  $p(x)$  in  $P$ , we have

$$\langle e^{yt} \mid p(x) \rangle = p(y) \quad (1.2)$$

and

$$e^{yt}p(x) = p(x+y) \quad (1.3)$$

cf. [6].

**Theorem 1.** ([6, p. 20, Theorem 2.3.6]) *Let  $f(t)$  be a delta series and let  $g(t)$  be an invertible series. Then there exist a unique sequence  $s_n(x)$  of polynomials satisfying the orthogonality conditions*

$$\langle g(t)f(t)^k \mid s_n(x) \rangle = n!\delta_{n,k} \quad (1.4)$$

for all  $n, k \geq 0$ .

The sequence  $s_n(x)$  in (1.4) is the Sheffer polynomials for pair  $(g(t), f(t))$ , where  $g(t)$  must be invertible and  $f(t)$  must be delta series. The Sheffer polynomials for pair  $(g(t), t)$  is the Appell polynomials or Appell sequences for  $g(t)$ .

The Appell polynomials are defined by means of the following generating function:

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}, \quad (1.5)$$

cf. [6].

The Appell polynomials satisfy the following relations:

$$s_n(x) = g(t)^{-1} x^n, \quad (1.6)$$

derivative formula

$$ts_n(x) = s'_n(x) = ns_{n-1}(x) \quad (1.7)$$

and

$$\frac{1}{t}s_n(x) = \frac{1}{n+1}s_{n+1}(x), \quad (1.8)$$

recurrence formula

$$s_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) s_n(x), \quad (1.9)$$

and multiplication formula, for  $\alpha \neq 0$

$$s_n(\alpha x) = \alpha^n \frac{g(t)}{g\left(\frac{t}{\alpha}\right)} s_n(x).$$

(see, for details, [6]; and see also [1], [3], [4]).

The remainder of this paper is organized as follows: We modify generating functions for the Milne-Thomson's polynomials  $\Phi_n^{(a)}(x)$ . We give some properties of this functions. By applying the Umbral algebra and Umbral calculus, we derive some identities related to Hermite polynomials, Bernoulli polynomials and Stirling numbers of second kind.

## 2. NEW TYPE POLYNOMIALS

We modify the Milne-Thomson's polynomials  $\Phi_n^{(a)}(x)$  (see for detail [5]) as  $\Phi_n^{(a)}(x, v)$  of degree  $n$  and order  $a$  by the means of the following generating function:

$$g_1(t, x; a, v) = f(t, a) e^{xt+h(t,v)} = \sum_{n=0}^{\infty} \Phi_n^{(a)}(x, v) \frac{t^n}{n!} \quad (2.1)$$

where  $f(t, a)$  is a function of  $t$  and the integer  $a$ .

Observe that  $\Phi_n^{(a)}(x, 0) = \Phi_n^{(a)}(x)$  cf. [5].

**Remark 1.** Setting  $f(t, a) = \left(\frac{t}{e^t-1}\right)^a$  in (2.1), we obtain the following polynomials by

$$g_2(t, x; a, v) = \left(\frac{t}{e^t-1}\right)^a e^{xt+h(t,v)} = \sum_{n=0}^{\infty} \beta_n^{(a)}(x; v) \frac{t^n}{n!}. \quad (2.2)$$

Observe that the polynomials  $\beta_n^{(a)}(x; v)$  are related to not only Bernoulli polynomials but also the Hermite polynomials. For example, if  $h(t, 0) = 0$  in (2.2), we have

$$\beta_n^{(a)}(x, 0) = B_n^{(a)}(x),$$

where  $B_n^{(a)}(x)$  denotes the Bernoulli polynomials of higher order which is, defined by means of the following generating function

$$f_B(t, x; a) = \left(\frac{t}{e^t-1}\right)^a e^{xt} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!}.$$

One can easily see that  $B_n^{(a)}(0) = B_n^{(a)}$ , that is

$$f_B(t; a) = \left(\frac{t}{e^t-1}\right)^a = \sum_{n=0}^{\infty} B_n^{(a)} \frac{t^n}{n!}.$$

If we take  $h(t) = -\frac{vt^2}{2}$  in (2.2), we have

$$\left(\frac{t}{e^t-1}\right)^a e^{xt-\frac{vt^2}{2}} = \sum_{n=0}^{\infty} \left({}_H\beta_n^{(a)}(x, v)\right) \frac{t^n}{n!}.$$

Hence, we get

$${}_H\beta_n^{(0)}(x, v) = H_n^{(v)}(x)$$

where  $H_n^{(v)}(x)$  denotes the Hermite polynomials of higher order, which is defined by means of the following generating function:

$$f_H(x, t; v) = e^{xt-\frac{vt^2}{2}} = \sum_{n=0}^{\infty} H_n^{(v)}(x) \frac{t^n}{n!}.$$

We define the following functional equation:

$$g_2(t, x; a, v) = f_B(t, x; a) e^{h(t,v)}. \quad (2.3)$$

By the above functional equation, we get

$$g_2(t, x; a, v) = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{h(t, v)^n}{n!}. \quad (2.4)$$

If we set  $h(t, v) = -vt$  in (2.4), we have

$$\beta_n^{(a)}(x, v) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_j^{(a)}(x) v^{n-j}.$$

We define the following functional equation:

$$g_2(t, x; a, v) = f_B(t; a) e^{xt+h(t,v)} \quad (2.5)$$

If we set  $h(t, v) = -\frac{v}{2}t^2$  in (2.3), we obtain the following theorem:

**Theorem 2.**

$$\beta_n^{(a)}(x, v) = \sum_{j=0}^n \binom{n}{j} B_j^{(a)}(x) H_{n-j}^{(v)}.$$

From (2.5), we get

$$\frac{\partial}{\partial x} g_2(t, x; a, v) = t g_2(t, x; a, v).$$

By using the above partial derivative equation, we obtain the following theorem:

**Theorem 3.**

$$\frac{\partial}{\partial x} \beta_n^{(a)}(x, v) = n \beta_{n-1}^{(a)}(x, v).$$

By using (1.7) and the above theorem, it is easily to see that  $\beta_n^{(a)}(x, v)$  is an Appell-type sequence.

### 3. SOME IDENTITIES FOR THE POLYNOMIALS ${}_H\beta_n^{(a)}(x, v)$

In this section, by applying the Umbral algebra and Umbral calculus, we derive some identities related to the polynomials  ${}_H\beta_n^{(a)}(x, v)$ .

By substituting

$$g(t) = \left( \frac{e^t - 1}{t} \right)^a e^{\frac{vt^2}{2}} \quad (3.1)$$

into (1.6), one can easily obtain the following lemma:

**Lemma 1.** *Let  $n \in \mathbb{N}_0$ . The following relationship holds true:*

$${}_H\beta_n^{(a)}(x, v) = \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{vt^2}{2}} x^n.$$

By using (1.7) and (1.8), we arrive at the following lemma:

**Lemma 2.**

$$t_H \beta_n^{(a)}(x, v) = n_H \beta_{n-1}^{(a)}(x, v), \quad (3.2)$$

and

$$\frac{1}{t_H} \beta_n^{(a)}(x, v) = \frac{1}{n+1_H} \beta_{n+1}^{(a)}(x, v). \quad (3.3)$$

The action of a linear operator  $(e^t - 1)$  on the polynomial  ${}_H \beta_n^{(a)}(x, v)$  is given by the following lemma:

**Lemma 3.**

$$(e^t - 1)_H \beta_n^{(a)}(x, v) = n_H \beta_{n-1}^{(a-1)}(x, v).$$

*Proof.* By using Lemma 1, we obtain

$$(e^t - 1)_H \beta_n^{(a)}(x, v) = (e^t - 1) \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{vt^2}{2}} x^n.$$

After some calculations in the above equation, we get

$$(e^t - 1)_H \beta_n^{(a)}(x, v) = t_H \beta_n^{(a-1)}(x, v).$$

Using (3.2) in the above equation, we arrive at the desired result.  $\square$

From Lemma 3, we arrive at the following result:

**Corollary 1.**

$$e_H^t \beta_n^{(a)}(x, v) = n_H \beta_{n-1}^{(a-1)}(x, v) + {}_H \beta_n^{(a)}(x, v). \quad (3.4)$$

**Theorem 4.**

$${}_H \beta_n^{(a)}(x + 1, v) = n_H \beta_{n-1}^{(a-1)}(x, v) + {}_H \beta_n^{(a)}(x, v).$$

*Proof.* Using (1.3), we get

$$e_H^t \beta_n^{(a)}(x, v) = {}_H \beta_n^{(a)}(x + 1, v).$$

Combining the above equation with (3.4), we complete the proof.  $\square$

By applying  $\frac{1}{e^t - 1}$  to the polynomial  ${}_H \beta_n^{(a)}(x, v)$ , we give the following lemma

**Lemma 4.**

$$\frac{1}{e^t - 1_H} \beta_n^{(a)}(x, v) = \frac{1}{n + 1_H} \beta_{n+1}^{(a+1)}(x, v).$$

*Proof.* From Lemma 1, we get

$$\frac{1}{e^t - 1_H} \beta_n^{(a)}(x, v) = \frac{1}{e^t - 1} \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{vt^2}{2}} x^n.$$

After some calculations, we obtain

$$\frac{1}{e^t - 1_H} \beta_n^{(a)}(x, v) = \frac{1}{t_H} \beta_n^{(a+1)}(x, v).$$

By using (3.3) in the above equation, we arrive at the desired result.  $\square$

**Theorem 5** (Recurrence formula).

$${}_H\beta_{n+1}^{(a)}(x, v) = \frac{1}{n-a+1} \left( (x-a)(n+1) {}_H\beta_n^{(a)}(x, v) - a {}_H\beta_{n+1}^{(a+1)}(x, v) - n(n+1) {}_H\beta_{n-1}^{(a)}(x, v) \right).$$

*Proof.* By using (3.1) into (1.9), we obtain

$${}_H\beta_{n+1}^{(a)}(x, v) = \left( x - \frac{ae^t}{e^t - 1} + \frac{a}{t} - t \right) \left( {}_H\beta_n^{(a)}(x, v) \right).$$

After elementary manipulations in this equation by using (3.2), (3.3), (3.4) and Lemma 4, we arrive at the last result.  $\square$

**Theorem 6.** Let  $k, a \in \mathbb{N}$  and  $k > a$ . We have

$$\left\langle (e^t - 1)^k \mid \left( {}_H\beta_n^{(a)}(x, v) \right) \right\rangle = \sum_{m=0}^{\infty} \frac{(-v)^{2m} (k-a)! (n)_{2m+a}}{(m!) 2^m} S(n-2m-a, k-a),$$

where  $S(n-2m-a, k-a)$  denotes the Stirling numbers of second kind.

*Proof.* Using Lemma 1, we get

$$\left\langle (e^t - 1)^k \mid \left( {}_H\beta_n^{(a)}(x, v) \right) \right\rangle = \left\langle (e^t - 1)^k \mid \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{vt^2}{2}} x^n \right\rangle.$$

By using (1.1), we obtain

$$\left\langle (e^t - 1)^k \mid \left( {}_H\beta_n^{(a)}(x, v) \right) \right\rangle = \left\langle (e^t - 1)^{k-a} \mid t^a e^{-\frac{vt^2}{2}} x^n \right\rangle.$$

After some calculations, we have

$$\left\langle (e^t - 1)^k \mid \left( {}_H\beta_n^{(a)}(x, v) \right) \right\rangle = \left\langle (e^t - 1)^{k-a} \mid \sum_{m=0}^{\infty} \frac{(-v)^{2m}}{m! 2^m} t^{2m+a} x^n \right\rangle.$$

Thus, using (3.2) in the above equation, we get

$$\left\langle (e^t - 1)^k \mid \left( {}_H\beta_n^{(a)}(x, v) \right) \right\rangle = \sum_{m=0}^{\infty} \frac{(-v)^{2m}}{(m!) 2^m} \left\langle (e^t - 1)^{k-a} \mid (n)_{2m+a} x^{n-2m-a} \right\rangle. \quad (3.5)$$

By substituting

$$S(n-2m-a, k-a) = \frac{1}{(k-a)!} \left\langle (e^t - 1)^{k-a} \mid x^{n-2m-a} \right\rangle$$

cf. [6, pp. 59] the above equation into (3.5), we obtain

$$\left\langle (e^t - 1)^k \mid \left( {}_H\beta_n^{(a)}(x, v) \right) \right\rangle = \sum_{m=0}^{\infty} \frac{(-v)^{2m} (k-a)! (n)_{2m+a}}{(m!) 2^m} S(n-2m-a, k-a).$$

$\square$

A relationship between  $B_n^{(a)}(x)$  and  ${}_H\beta_n^{(a)}(x, v)$  is given by the following theorem:

**Theorem 7.** The following relationship holds true:

$$e^{-\frac{vt^2}{2}} B_n^{(a)}(x) = {}_H\beta_n^{(a)}(x, v).$$

*Proof.* Setting

$$B_n^{(a)}(x) = \left( \frac{t}{e^t - 1} \right)^a x^n,$$

we obtain,

$$e^{-\frac{vt^2}{2}} B_n^{(a)}(x) = e^{-\frac{vt^2}{2}} \left( \frac{t}{e^t - 1} \right)^a x^n.$$

Using Lemma 1, we arrive at the final result. □

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